Linear Algebra and its Applications 436 (2012) 2105–2116

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Orthogonal polynomials, Catalan numbers, and a general Hankel determinant evaluation

Wathek Chammam [b], Francisco Marcellán [a,*], Ridha Sfaxi [b]

[a] Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911 Leganés, Spain
[b] Faculté des Sciences de Gabès, Département de Mathématiques, Cité Erriadh, 6072 Gabès, Tunisia

ARTICLE INFO

Article history:
Received 11 January 2011
Accepted 10 October 2011
Available online 9 November 2011

Submitted by R.A. Brualdi

AMS classification:
05A10
33C45
11C20
11B65

Keywords:
Catalan numbers
Hankel transform
Orthogonal polynomials

ABSTRACT

In this paper we deal with Hankel determinants of the form $\det[a_{i+j+r}(x)]_{i,j=0}^n$, where r is a non-negative integer, $a_{n+r}(x) = a_{n+r} + a_{n+r-1}x + \cdots + a_0x^{n+r}$ and $(a_n)_{n\geqslant 0}$ is a sequence complex numbers. When $a_0 \neq 0$ and the Hankel determinants associated with the sequence $(a_{n+r+1})_{n\geqslant 0}$ are not identically zero, we show that $(\det[a_{i+j+r}(x)]_{i,j=0}^n)_{n\geqslant 0}$ is a sequence of polynomials satisfying a three-term recurrence relation. We illustrate our result by evaluating the Hankel determinant associated with the sequence $\det\left[\sum_{\nu=0}^{l+k+r} \frac{1}{l+k+r+1-\nu}\binom{2(l+k+r-\nu)}{l+k+r-\nu} x^\nu\right]_{l,k=0}^n$, for $r = 0$ and $r = 1$.

1. Introduction

Let $A := (a_n)_{n\geqslant 0}$ be a sequence of complex numbers. For any integer $r \geqslant 0$, we set $A^{(r)} := (a_{n+r})_{n\geqslant 0}$. By convention, $A^{(0)} = A$. The *Hankel transform* of the sequence $A^{(r)}$ is the sequence of the so-called shifted Hankel determinants $H_0(A^{(r)}), H_1(A^{(r)}), \ldots$, (see [7]) given by

$$H_n(A^{(r)}) = \det[a_{l+k+r}]_{l,k=0}^n. \tag{1.1}$$

Note that these are determinants of $(n+1) \times (n+1)$ matrices.

Since the 19th century, the evaluation of Hankel determinants has attracted attention as one of the most interesting topics in the framework of the moment theory and orthogonal polynomials. In the

* Corresponding author.
E-mail addresses: wathek1@ymail.com (W. Chammam), pacomarc@ing.uc3m.es (F. Marcellán), Ridha.Sfaxi@isggb.rnu.tn (R. Sfaxi).

0024-3795/$ - see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2011.10.010

literature (see [5,6]), several techniques for the evaluation of many classes of Hankel determinants are described. One also can find there an extensive bibliography on the subject.

The aim of our contribution is focused on the evaluation of the Hankel determinants with a certain class of monic polynomials as entries. The basic idea of this work comes from [11], where the author evaluates the Hankel determinant associated with the Catalan numbers by applying the technique of γ-operator on the Hankel determinant associated with a particular sequence of monic polynomials satisfying a two-term recurrence relation. For more information on the technique of γ-operator, see [11,12].

We derive an explicit formula for this class of Hankel determinants using another approach based on some elementary properties of determinants and the theory of orthogonal polynomials.

2. Preliminaries

Let $\mathbb{P}$ be the linear space of polynomials in one variable with complex coefficients and $\mathbb{P}'$ its dual space. We denote by $\langle \mathscr{U}, p\rangle$ the action of $\mathscr{U} \in \mathbb{P}'$ on $p \in \mathbb{P}$ and by $(\mathscr{U})_n := \langle \mathscr{U}, x^n\rangle$, $n \geqslant 0$, the sequence of moments of $\mathscr{U}$ with respect to the polynomial sequence $\{x^n\}_{n\geqslant 0}$. Let us define the following operations in $\mathbb{P}'$. For linear functionals $\mathscr{U}$ and $\mathscr{V}$, any polynomial q, and any $(a, b, c) \in \mathbb{C}^* \times \mathbb{C}^2$, let $D\mathscr{U} = \mathscr{U}'$, $q\mathscr{U}$, $(x-c)^{-1}\mathscr{U}$, $\tau_{-b}\mathscr{U}$, $h_a\mathscr{U}$, and $\mathscr{U}\mathscr{V}$ be the linear functionals defined by duality (see [3,8,9])

$$\langle \mathscr{U}', p\rangle := -\langle \mathscr{U}, p'\rangle,$$
$$\langle q\mathscr{U}, p\rangle := \langle \mathscr{U}, qp\rangle,$$
$$\langle (x-c)^{-1}\mathscr{U}, p\rangle := \langle \mathscr{U}, \theta_c p\rangle = \left\langle \mathscr{U}, \frac{p(x)-p(c)}{x-c}\right\rangle,$$
$$\langle \tau_{-b}\mathscr{U}, p\rangle := \langle \mathscr{U}, \tau_b p\rangle = \langle \mathscr{U}, p(x-b)\rangle,$$
$$\langle h_a\mathscr{U}, p\rangle := \langle \mathscr{U}, h_a p\rangle = \langle \mathscr{U}, f(ax)\rangle,$$
$$\langle \mathscr{U}\mathscr{V}, p\rangle := \langle \mathscr{U}, \mathscr{V}p\rangle, \quad p \in \mathbb{P},$$

where the right-multiplication of $\mathscr{V}$ by a polynomial p is a polynomial given by

$$(\mathscr{V}p)(x) := \left\langle \mathscr{V}_y, \frac{xp(x)-yp(y)}{x-y}\right\rangle, \quad p \in \mathbb{P}.$$

Notice that $\deg(\mathscr{V}p) = \deg p$ if and only if $(\mathscr{V})_0 \neq 0$.

We can associate with the sequence $A = (a_n)_{n\geqslant 0}$ a unique linear functional $\mathscr{U} \in \mathbb{P}'$ by setting

$$(\mathscr{U})_n = a_n, \quad n \geqslant 0. \tag{2.1}$$

The linear functional $\mathscr{U}$ is said to be quasi-definite if $H_n(A) \neq 0$ for every integer $n \geqslant 0$, [3]. In this case, there exists a unique sequence of monic polynomials (SMP) $(P_n)_{n\geqslant 0}$, i.e., $P_n(x) = x^n +$ lower degree terms, such that

(i) $\langle \mathscr{U}, x^\nu P_n\rangle = 0$, $\nu = 0, 1, \ldots, n-1$.
(ii) $\langle \mathscr{U}, x^n P_n\rangle \neq 0$.

$(P_n)_{n\geqslant 0}$ is said to be the sequence of monic orthogonal polynomials (SMOP) with respect to $\mathscr{U}$. It is very well known that $P_n(x)$ can be represented as a determinant (see [3])

$$P_0(x) = 1, \quad P_n(x) = \frac{(-1)^n}{H_{n-1}(A)} \begin{vmatrix} 1 & x & \ldots & x^n \\ a_0 & a_1 & \ldots & a_n \\ a_1 & a_2 & \ldots & a_{n+1} \\ \vdots & \vdots & & \vdots \\ a_{n-1} & a_n & \ldots & a_{2n-1} \end{vmatrix}, \quad n \geqslant 1. \tag{2.2}$$

The orthogonality of $(P_n)_{n\geqslant 0}$ can be characterized by a three-term recurrence relation (TTRR), according to the Favard's theorem (see [3, p. 21]),

$$\begin{cases} P_{-1}(x) &= 0, \quad P_0(x) = 1, \\ P_{n+1}(x) &= (x-\beta_n)P_n(x) - \gamma_n P_{n-1}(x), \quad n \geqslant 0, \end{cases} \tag{2.3}$$

where $\beta_n \in \mathbb{C}$, $\gamma_n \in \mathbb{C}^*$ for every integer $n \geqslant 0$, and $\gamma_0 = (\mathscr{U})_0$.

Furthermore, from the general theory of orthogonal polynomials (see [9])

$$\beta_n = \frac{\langle \mathscr{U}, xP_n^2\rangle}{\langle \mathscr{U}, P_n^2\rangle}, \quad \gamma_{n+1} = \frac{\langle \mathscr{U}, P_{n+1}^2\rangle}{\langle \mathscr{U}, P_n^2\rangle}, \quad n \geqslant 0, \tag{2.4}$$

$$\langle \mathscr{U}, P_n^2\rangle = \prod_{\nu=0}^{n} \gamma_\nu = \frac{H_n(A)}{H_{n-1}(A)}, \quad n \geqslant 0, \qquad H_{-1}(A) = 1. \tag{2.5}$$

We can associate to the sequence $A = (a_n)_{n\geqslant 0}$ a sequence of polynomials $(a_n(x))_{n\geqslant 0}$ given by [11]

$$a_n(x) := \sum_{\nu=0}^{n} a_{n-\nu} x^\nu, \quad n \geqslant 0. \tag{2.6}$$

Notice that the polynomials $a_n(x)$ satisfy

$$a_{-1}(x) = 0, \quad a_n(x) = xa_{n-1}(x) + a_n, \quad n \geqslant 0. \tag{2.7}$$

Thus, $\deg a_n(x) = n$ if and only if $a_0 \neq 0$.

In this paper we investigate the Hankel transform of the sequence of polynomials $(a_{n+r}(x))_{n\geqslant 0}$ where r is a fixed non-negative integer. Indeed, we define the sequence of determinants $(H_{n+r}(\binom{A}{r}| \, x))_{n\geqslant 0}$, where

$$H_{n+r}(\tbinom{A}{r}| \, x) := \det\left[a_{l+k+r}(x)\right]_{l,k=0}^{n}, \quad n \geqslant 0. \tag{2.8}$$

As a convention, for $r = 0$ we write $H_n(A \mid x) = H_n(\binom{A}{0}| \, x)$.

From $a_{n+r}(0) = a_{n+r}$ we get $H_{n+r}(\binom{A}{r}| \, 0) = H_n(A^{(r)})$.

For $A = C$, let $C_n = \binom{2n}{n}/(n+1)$ denote the nth Catalan number. In [11] the γ-operator technique is introduced to obtain a differential equation satisfied by $H_n(C \mid x)$ and, as a consequence, the following evaluation is proved

$$H_n(C \mid x) = \sum_{\nu=0}^{n} (-1)^\nu \binom{n+\nu}{n-\nu} x^\nu.$$

In [11], it is also shown that $H_n(C \mid x)$ has n real simple zeros on the interval $(0, 4)$. Furthermore, $(H_n(C \mid x))_{n\geqslant 0}$ is a Sturm sequence whose zeros satisfy the interlacing property. Here, we will show that $(H_n(C \mid x))_{n\geqslant 0}$ is a SMOP with respect to a positive-definite linear functional.

When $a_0 \neq 0$ and the linear functional $x^{r+1}\mathscr{U}$ is quasi-definite, i.e., $H_n(A^{(r+1)}) \neq 0$, for all $n \geqslant 0$, in Corollary 1 we will prove that $H_{n+r}(\binom{A}{r}| \, x)$ is a polynomial with $\deg H_{n+r}(\binom{A}{r}| \, x) = n+r$. Furthermore, the sequence of monic polynomials defined by $((-1)^n(a_0 H_{n-1}(A^{(r+1)}))^{-1} H_{n+r}(\binom{A}{r}| \, x))_{n\geqslant 1}$ satisfies a three-term recurrence relation. In particular, we prove that $((-1)^n(a_0 H_{n-1}(A^{(r+1)}))^{-1} H_{n+r}(\binom{A}{r}| \, x))_{n\geqslant 1}$ is a subsequence of a sequence of monic orthogonal polynomials if and only if either $r = 0$ or $r = 1$.

Indeed, for $r = 0$ the SMP $(\hat{H}_n(A \mid x))_{n\geqslant 0}$ defined by $\hat{H}_0(A \mid x) = 1$ and $\hat{H}_n(A \mid x) = (-1)^n(a_0 H_{n-1}(A^{(1)}))^{-1} H_n(A \mid x)$, $n \geqslant 1$, is a SMOP. More precisely, it is a co-recursive SMOP with respect to the linear functional $x\mathscr{U}$. On the other hand, for $r = 1$ the SMP $(\hat{H}_n(\binom{A}{1}| \, x))_{n\geqslant 0}$ where $\hat{H}_0(\binom{A}{1}| \, x) = 1$, $\hat{H}_1(\binom{A}{1}| \, x) = x + a_0^{-1}a_1$ and $\hat{H}_n(\binom{A}{1}| \, x) = (-1)^n(a_0 H_{n-1}(A^{(2)}))^{-1} H_{n+1}(\binom{A}{1}| \, x)$, $n \geqslant 1$, is also a SMOP.

The paper is organized as follows. In Section 2, the proofs of our main results are given. We obtain (see Theorem 3) a first evaluation of the Hankel determinants $H_{n+r}(\binom{A}{r}|\ x)$ in the general setting. From there, the evaluation of these Hankel determinants (see Corollary 1) becomes more useful and feasible, assuming the linear functional is quasi-definite. In Section 3, we focus our attention on the orthogonality of the polynomial sequence $((-1)^n(a_0H_{n-1}(A^{(r+1)}))^{-1}H_{n+r}(\binom{A}{r}|\ x))_{n\geqslant 1}$, where $r=0$ or $r=1$. Finally, some illustrative examples based on the Catalan numbers are presented.

3. Main results

Using (1.8), elementary row operations yield

$$H_{n+r}(\binom{A}{r}|\ x)=\begin{vmatrix} a_r(x) & a_{r+1}(x) & \dots & a_{n+r}(x)\\ a_{r+1} & a_{r+2} & \dots & a_{n+r+1}\\ a_{r+2} & a_{r+3} & \dots & a_{n+r+2}\\ \vdots & \vdots & & \vdots\\ a_{n+r} & a_{n+r+1} & \dots & a_{2n+r}\end{vmatrix},\quad n\geqslant 1. \tag{3.1}$$

Expanding the above determinant by the first row, we get

$$H_{n+r}(\binom{A}{r}|\ x)=\sum_{j=0}^{n}(-1)^jD_{j+1,r}a_{j+r}(x), \tag{3.2}$$

where $D_{j,r}$ denotes the determinant of the $n\times n$ matrix obtained by deleting the jth column of the $n\times(n+1)$ matrix

$$\begin{bmatrix} a_{r+1} & a_{r+2} & \dots & a_{n+r+1}\\ a_{r+2} & a_{r+3} & \dots & a_{n+r+2}\\ \vdots & \vdots & & \vdots\\ a_{n+r} & a_{n+r+1} & \dots & a_{2n+r}\end{bmatrix},\quad n\geqslant 1. \tag{3.3}$$

Notice that $D_{n+1,r}=H_{n-1}(A^{(r+1)})$.

From (1.7) and (2.2), we get the following result.

Lemma 1. *Let $A=(a_n)_{n\geqslant 0}$ be a sequence of complex numbers and let $\mathscr{U}\in\mathbb{P}'$ be such that $(\mathscr{U})_n=a_n$, $n\geqslant 0$. The following statements are equivalent.*

(a) $\deg H_{n+r}(\binom{A}{r}|\ x)=n+r,\ n\geqslant 0$.
(b) $a_0\neq 0$ *and* $H_{n-1}(A^{(r+1)})\neq 0,\ n\geqslant 1$.
(c) $a_0\neq 0$ *and* $x^{r+1}\mathscr{U}$ *is quasi-definite.*

The next lemma will play an important role in the sequel.

Lemma 2. *Let $A=(a_n)_{n\geqslant 0}$ be a sequence of complex numbers and let $\mathscr{U}\in\mathbb{P}'$ be such that $(\mathscr{U})_n=a_n$, $n\geqslant 0$. Suppose $(a_n(x))_{n\geqslant 0}$ is defined as in (1.7). Then*

$$a_n(x)=(\mathscr{U}x^n)(x),\quad n\geqslant 0. \tag{3.4}$$

Proof. From (1.2) and (1.7), we obtain

$$(\mathscr{U}x^n)(x)=\left\langle \mathscr{U}_y,\frac{x^{n+1}-y^{n+1}}{x-y}\right\rangle=\sum_{\nu=0}^{n}(\mathscr{U})_{n-\nu}x^\nu=\sum_{\nu=0}^{n}a_{n-\nu}x^\nu=a_n(x),\quad n\geqslant 0.$$

Thus, our statement holds. □

As a straightforward consequence of Lemma 2 and (2.1) the following expression of the Hankel determinants $H_n(^A_r|\,x)$ holds.

Theorem 3. *Let $A=(a_n)_{n\geqslant 0}$ be a sequence of complex numbers and let $\mathscr{U}\in\mathbb{P}'$ be such that $(\mathscr{U})_n=a_n$, $n\geqslant 0$. For any integer $r\geqslant 0$, we have*

$$H_{n+r}(^A_r|\,x)=\mathscr{U}x^rQ_n(x;A^{(r+1)}),\quad n\geqslant 0, \tag{3.5}$$

where $$Q_0(x;A^{(r+1)})=1,\quad Q_n(x;A^{(r+1)})=\begin{vmatrix} 1 & x & \dots & x^n \\ a_{r+1} & a_{r+2} & \dots & a_{n+r+1} \\ a_{r+2} & a_{r+3} & \dots & a_{n+r+2} \\ \vdots & \vdots & & \vdots \\ a_{n+r} & a_{n+r+1} & \dots & a_{2n+r} \end{vmatrix},\ n\geqslant 1.$$

According to (1.3), if $H_{n-1}(A^{(r+1)})\neq 0$, $n\geqslant 1$, then the sequence of monic polynomials defined by $((-1)^nH_{n-1}(A^{(r+1)})^{-1}Q_n(x;A^{(r+1)}))_{n\geqslant 0}$ is orthogonal with respect to the linear functional $x^{r+1}\mathscr{U}$. Furthermore,

$$\hat{Q}_n(x;A^{(r+1)}):=\frac{(-1)^n}{H_{n-1}(A^{(r+1)})}Q_n(x;A^{(r+1)}),\quad n\geqslant 0, \tag{3.6}$$

then, the SMOP $(\hat{Q}_n(x;A^{(r+1)}))_{n\geqslant 0}$ satisfies the TTRR,

$$\begin{cases} \hat{Q}_{-1}(x;A^{(r+1)}) &= 0,\quad \hat{Q}_0(x;A^{(r+1)})=1,\\ \hat{Q}_{n+1}(x;A^{(r+1)}) &= (x-\alpha_n^{(r+1)})\hat{Q}_n(x;A^{(r+1)})-\\ & \qquad\qquad \lambda_n^{(r+1)}\hat{Q}_{n-1}(x;A^{(r+1)}),\quad n\geqslant 0, \end{cases} \tag{3.7}$$

where $\alpha_n^{(r+1)}\in\mathbb{C}$, $\lambda_n^{(r+1)}\in\mathbb{C}^*$, and $\lambda_0^{(r+1)}=a_{r+1}$.

Assuming $a_0\neq 0$ and $H_{n-1}(A^{(r+1)})\neq 0$, $n\geqslant 1$, the polynomials $\hat{H}_{n+r}(^A_r|\,x)$, $n\geqslant 1$, given by

$$\hat{H}_{n+r}(^A_r|\,x):=\frac{(-1)^n}{a_0H_{n-1}(A^{(r+1)})}H_{n+r}(^A_r|\,x)=a_0^{-1}\mathscr{U}x^r\hat{Q}_n(x;A^{(r+1)}), \tag{3.8}$$

are monic and $\deg H_{n+r}(^A_r|\,x)=n+r$.

Under the above assumptions, we get the following:

Corollary 1. *Let $A=(a_n)_{n\geqslant 0}$ be a sequence of complex numbers and let $\mathscr{U}\in\mathbb{P}'$ be such that $(\mathscr{U})_n=a_n$, $n\geqslant 0$. Suppose that $a_0\neq 0$ and the linear functional $x^{r+1}\mathscr{U}$ is quasi-definite. Then the SMP $(\hat{H}_{n+r}(^A_r|\,x))_{n\geqslant 1}$ given by*

$$\hat{H}_{n+r}(^A_r|\,x)=\frac{(-1)^n}{a_0\det[a_{l+k+r+1}]_{l,k=0}^{n-1}}\det\left[\sum_{\nu=0}^{l+k+r}a_{l+k+r-\nu}x^\nu\right]_{l,k=0}^{n},\quad n\geqslant 1,$$

satisfies the following three term recurrence relation (TTRR)

$$\begin{cases} \hat{H}_r(^A_r|\,x)=a_0^{-1}\mathscr{U}x^r,\\ \hat{H}_{r+1}(^A_r|\,x)=(x-\alpha_0^{(r+1)})\hat{H}_r(^A_r|\,x)+a_0^{-1}a_{r+1},\\ \hat{H}_{n+r+1}(^A_r|\,x)=(x-\alpha_n^{(r+1)})\hat{H}_{n+r}(^A_r|\,x)-\lambda_n^{(r+1)}\hat{H}_{n+r-1}(^A_r|\,x),\quad n\geqslant 1, \end{cases} \tag{3.9}$$

where $\alpha_n^{(r+1)}$ and $\lambda_n^{(r+1)}$ are the coefficients in the TTRR for the SMOP with respect to $x^{r+1}\mathscr{U}$.

Proof. First, we need the following formula (see [9])

$$(\mathscr{U}xp)(x) = x(\mathscr{U}p)(x) + \langle y\mathscr{U}_y, p(y)\rangle, \quad p \in \mathbb{P}. \tag{3.10}$$

From (2.7), (2.8), and (2.10) with $p(x) = x^r\hat{Q}_n(x, A^{(r+1)})$, we get

$$\hat{H}_{n+r+1}(\begin{smallmatrix}A\\r\end{smallmatrix}|\ x) = (x - \alpha_n^{(r+1)})\hat{H}_{n+r}(\begin{smallmatrix}A\\r\end{smallmatrix}|\ x) - \lambda_n^{(r+1)}\hat{H}_{n+r-1}(\begin{smallmatrix}A\\r\end{smallmatrix}|\ x) + a_0^{-1}\langle y^{r+1}\mathscr{U}_y, \hat{Q}_n(y, A^{(r+1)})\rangle, \quad n \geqslant 2.$$

Since $(\hat{Q}_n(x, A^{(r+1)}))_{n\geqslant 0}$ is a SMOP with respect to $x^{r+1}\mathscr{U}$, then $\langle y^{r+1}\mathscr{U}_y, \hat{Q}_n(y, A^{(r+1)})\rangle = a_{r+1}\delta_{n,0}$, $n \geqslant 0$. Therefore,

$$\hat{H}_{n+r+1}(\begin{smallmatrix}A\\r\end{smallmatrix}|\ x) = (x - \alpha_n^{(r+1)})\hat{H}_{n+r}(\begin{smallmatrix}A\\r\end{smallmatrix}|\ x) - \lambda_n^{(r+1)}\hat{H}_{n+r-1}(\begin{smallmatrix}A\\r\end{smallmatrix}|\ x), \quad n \geqslant 2,$$

By adding $\hat{H}_r(\begin{smallmatrix}A\\r\end{smallmatrix}|\ x) = a_0^{-1}\mathscr{U}x^r$ and, proceeding as above, we get $\hat{H}_{r+1}(\begin{smallmatrix}A\\r\end{smallmatrix}|\ x) = (x - \alpha_0^{(r+1)})\hat{H}_r(\begin{smallmatrix}A\\r\end{smallmatrix}|\ x) + a_0^{-1}a_{r+1}$. $\square$

4. Special cases and illustrative examples

From Favard's Theorem and (2.9) , we can deduce that for $r \geqslant 2$ the SMP $(\hat{H}_n(\begin{smallmatrix}A\\r\end{smallmatrix}|\ x))_{n\geqslant r}$ is not a subsequence of a SMOP. Whereas for $r = 0, 1$ we will show that $\{\hat{H}_n(\begin{smallmatrix}A\\r\end{smallmatrix}|\ x)\}_{n\geqslant r}$ is a subsequence of a SMOP $\{\hat{H}_n(\begin{smallmatrix}A\\r\end{smallmatrix}|\ x)\}_{n\geqslant 0}$.

4.1. *Evaluation of* $\det[\sum_{\nu=0}^{l+k} a_{l+k-\nu}x^\nu]_{l,k=0}^n$

Let us take $r = 0$ in Corollary 1 and assume $\hat{H}_0(A \mid x) = 1$. Then, the SMP $(\hat{H}_n(A \mid x))_{n\geqslant 0}$ satisfies the following TTRR,

$$\begin{cases} \hat{H}_0(A \mid x) &= 1, \quad \hat{H}_1(A \mid x) = x - \alpha_0^{(1)} + a_0^{-1}a_1, \\ \hat{H}_{n+1}(A \mid x) &= (x - \alpha_n^{(1)})\hat{H}_n(A \mid x) - \lambda_n^{(1)}\hat{H}_{n-1}(A \mid x), \quad n \geqslant 1. \end{cases} \tag{4.1}$$

So, the SMP $(\hat{H}_n(A \mid x))_{n\geqslant 0}$ is the co-recursive of the SMOP $(\hat{Q}_n(x, A^{(1)}))_{n\geqslant 0}$, since it is generated by the TTRR (2.7) with $r = 0$, where $\alpha_0^{(1)}$ is replaced by $\alpha_0^{(1)} - a_1a_0^{-1}$. For more information, see [3,9]. Denoting by $(\hat{Q}_n^{(1)}(x, A^{(1)}))_{n\geqslant 0}$ the first kind associated SMOP of the sequence $(\hat{Q}_n(x, A^{(1)}))_{n\geqslant 0}$ that is defined by the following TTRR (see [3])

$$\begin{cases} \hat{Q}_{-1}^{(1)}(x, A^{(1)}) = 0, \quad \hat{Q}_0^{(1)}(x, A^{(1)}) = 1, \\ \hat{Q}_{n+1}^{(1)}(x, A^{(1)}) = (x - \alpha_{n+1}^{(1)})\hat{Q}_n^{(1)}(x, A^{(1)}) - \lambda_{n+1}^{(1)}\hat{Q}_{n-1}^{(1)}(x, A^{(1)}), \quad n \geqslant 0, \end{cases} \tag{4.2}$$

we get

$$\hat{H}_n(A \mid x) = \hat{Q}_n(x, A^{(1)}) + a_1a_0^{-1}\hat{Q}_{n-1}^{(1)}(x, A^{(1)}), \quad n \geqslant 0. \tag{4.3}$$

From (3.3) and Corollary 1 with $r = 0$, the following evaluation of the Hankel determinant $\det[\sum_{\nu=0}^{l+k} a_{l+k-\nu}x^\nu]_{l,k=0}^n$ holds.

Corollary 2. *Let $A = (a_n)_{n\geqslant 0}$ be a sequence of complex numbers and let $\mathscr{U} \in \mathbb{P}'$ be such that $(\mathscr{U})_n = a_n$, $n \geqslant 0$. Assuming that $a_0 \neq 0$ and the linear functional $x\mathscr{U}$ is quasi-definite, if $(\hat{Q}_n(x, A^{(1)}))_{n\geqslant 0}$ is the corresponding SMOP, then*

$$\det\left[\sum_{\nu=0}^{l+k} a_{l+k-\nu}x^\nu\right]_{l,k=0}^n = a_0(-1)^n \det[a_{l+k+1}]_{l,k=0}^{n-1}\hat{Q}_n^*(x, A^{(1)}), \quad n \geqslant 1,$$

where $(\hat{Q}_n^*(x, A^{(1)}))_{n\geqslant 0}$ *is the co-recursive SMOP of the SMOP* $(\hat{Q}_n(x, A^{(1)}))_{n\geqslant 0}$*, given by*

$$\hat{Q}_n^*(x, A^{(1)}) = \hat{Q}_n(x, A^{(1)}) + a_0^{-1}a_1\hat{Q}_{n-1}^{(1)}(x, A^{(1)}), \quad n \geqslant 0.$$

As an application of the previous result, we will recover (see [11]) the evaluation of $H_n(C \mid x) = \det\left[\sum_{\nu=0}^{l+k}\frac{1}{l+k+1-\nu}\binom{2(l+k)-2\nu}{l+k-\nu}x^{\nu}\right]_{l,k=0}^{n}$.

First, let us take in Corollary 2, $A = C$ where $C_n = \binom{2n}{n}/(n+1)$, the nth Catalan number. Notice that $(n+2)C_{n+1} = (4n+2)C_n,\ n \geqslant 0$. So, the related linear functional $\mathscr{U}$ with $(\mathscr{U})_n = C_n,\ n \geqslant 0$, is a solution of the functional equation

$$\big(x(x-4)\mathscr{U}\big)' + 2(-x+1)\mathscr{U} = 0. \tag{4.4}$$

According to [8,10], $\mathscr{U} = h_{-2} \circ \tau_{-1}\mathcal{J}(\frac{1}{2}, -\frac{1}{2})$, where $\mathcal{J}(\frac{1}{2}, -\frac{1}{2})$ is the Jacobi monic linear functional with parameters $\alpha = -\beta = 1/2$, i.e., Chebyshev linear functional of the third kind.

In the same way, $\mathscr{U}_1 := x\mathscr{U}$ is monic (i.e., $(\mathscr{U}_1)_0 = 1$) and positive-definite. Indeed, according to (3.4) $\mathscr{U}_1$ satisfies

$$\big(x(x-4)\mathscr{U}_1\big)' + 3(-x+2)\mathscr{U}_1 = 0. \tag{4.5}$$

From [10], $\mathscr{U}_1 = h_{-2} \circ \tau_{-1}\mathcal{J}(\frac{1}{2}, \frac{1}{2})$, where $\mathcal{J}(\frac{1}{2}, \frac{1}{2})$ is the Chebyshev linear functional of the second kind.

Here, the SMOP $(\hat{Q}_n(x, C^{(1)}))_{n\geqslant 0}$ with respect to $\mathscr{U}_1$ is given by

$$\hat{Q}_n(x, C^{(1)}) = (-2)^n\hat{U}_n\left(\frac{2-x}{2}\right), \quad n \geqslant 0, \tag{4.6}$$

where $(\hat{U}_n(x))_{n\geqslant 0}$ is the Chebyshev SMOP of second kind.

Using the TTRR satisfied by $(\hat{U}_n(x))_{n\geqslant 0}$ (see [10]) we get the following TTRR satisfied by $(\hat{Q}_n(x, C^{(1)}))_{n\geqslant 0}$,

$$\begin{cases}\hat{Q}_{-1}(x, C^{(1)}) &= 0, \quad \hat{Q}_0(x, C^{(1)}) = 1,\\ \hat{Q}_{n+1}(x, C^{(1)}) &= (x-2)\hat{Q}_n(x, C^{(1)}) - \hat{Q}_{n-1}(x, C^{(1)}), \quad n \geqslant 0.\end{cases} \tag{4.7}$$

From (3.7) and (1.6), we recover the well-known Hankel determinant evaluation [1,4],

$$\det[C_{l+k+1}]_{l,k=0}^{n} = 1, \quad n \geqslant 0. \tag{4.8}$$

Using again (3.7) we obtain the following relation:

$$\hat{Q}_n^{(1)}(x, C^{(1)}) = \hat{Q}_n(x, C^{(1)}), \quad n \geqslant 0. \tag{4.9}$$

From Corollary 2 with $A = C$ and by taking into account (3.6), (3.8), and (3.9) we obtain

$$\det\left[\sum_{\nu=0}^{l+k} C_{l+k-\nu}x^{\nu}\right]_{l,k=0}^{n} = 2^n\hat{U}_n\left(\frac{2-x}{2}\right) - 2^{n-1}\hat{U}_{n-1}\left(\frac{2-x}{2}\right), \quad n \geqslant 1. \tag{4.10}$$

By (3.3) with $A = C$ and (3.9), the SMOP $(\hat{H}_n(C \mid x))_{n\geqslant 0}$ and $(\hat{Q}_n(x, C^{(1)}))_{n\geqslant 0}$ verify the following connection relation:

$$\hat{H}_n(C \mid x) = \hat{Q}_n(x, C^{(1)}) + \hat{Q}_{n-1}(x, C^{(1)}), \quad n \geqslant 0.$$

From (3.1) with $A = C$, we get the TTRR satisfied by $(\hat{H}_n(C \mid x))_{n\geqslant 0}$,

$$\begin{cases}\hat{H}_0(C \mid x) = 1, \quad \hat{H}_1(C \mid x) = x - 1,\\ \hat{H}_{n+1}(C \mid x) = (x-2)\hat{H}_n(C \mid x) - \hat{H}_{n-1}(C \mid x), \quad n \geqslant 1.\end{cases}$$

From [10], we get $\hat{H}_n(C \mid x) = 2^n \hat{J}_n\left(\frac{x-2}{2}; -\frac{1}{2}, \frac{1}{2}\right)$, $n \geqslant 0$, where $\left(\hat{J}_n(x; -\frac{1}{2}, \frac{1}{2})\right)_{n\geqslant 0}$ is the Chebyshev SMOP of the fourth kind. Hence, the following evaluation holds

$$\det\left[\sum_{\nu=0}^{l+k} C_{l+k-\nu} x^{\nu}\right]_{l,k=0}^{n} = (-1)^n \hat{J}_n\left(\frac{x-2}{2}; -\frac{1}{2}, \frac{1}{2}\right), \quad n \geqslant 0. \tag{4.11}$$

Notice that $(\hat{H}_n(C \mid x))_{n\geqslant 0}$ is a SMOP with respect to a positive definite linear functional. Thus their zeros satisfy the interlacing property (see [3]).

Based on some properties of the Chebyshev SMOP of the second kind $(\hat{U}_n(x))_{n\geqslant 0}$, we can prove the following result.

Lemma 4

$$\det\left[\sum_{\nu=0}^{l+k} C_{l+k-\nu} x^{\nu}\right]_{l,k=0}^{n} = \frac{\cos\left(\left(n+\frac{1}{2}\right)z\right)}{\cos\left(\frac{z}{2}\right)}, \tag{4.12}$$

where $x = 2(1 - cos(z))$. *On the other hand,*

$$\det\left[\sum_{\nu=0}^{l+k} C_{l+k-\nu} x^{\nu}\right]_{l,k=0}^{n} = (-1)^n \prod_{\nu=0}^{n-1}\left(x - 4\sin^2\left[\frac{(2\nu+1)}{2n+1}\frac{\pi}{2}\right]\right), \quad n \geqslant 1. \tag{4.13}$$

Proof. The Chebyshev polynomials of second kind are defined by (see [2]),

$$\hat{U}_n(\cos(z)) = \frac{\sin((n+1)z)}{2^n \sin(z)}, \quad n \geqslant 0.$$

Letting $x = 2(1 - \cos(z))$ into the right-hand side of (3.10), we get

$$\begin{aligned}\det\left[\sum_{\nu=0}^{l+k} C_{l+k-\nu} x^{\nu}\right]_{l,k=0}^{n} &= \frac{\sin((n+1)z) - \sin(nz)}{\sin(z)} \\ &= \frac{\cos\left(\left(n+\frac{1}{2}\right)z\right)}{\cos\left(\frac{z}{2}\right)}.\end{aligned}$$

Hence (3.12) holds.

Since $z_\nu = \frac{(2\nu+1)\pi}{2n+1}$, $\nu = 0, 1, \ldots, n-1$, are the zeros of $\cos((n+\frac{1}{2})z)$ then $x_\nu = 2(1 - \cos(\frac{(2\nu+1)\pi}{2n+1})) = 4\sin^2(\frac{(2\nu+1)}{2n+1}\frac{\pi}{2})$ are the zeros of the polynomial $2^n\hat{U}_n(\frac{2-x}{2}) - 2^{n-1}\hat{U}_{n-1}(\frac{2-x}{2})$. Taking into account the degree of this polynomial is n, then

$$2^n\hat{U}_n\left(\frac{2-x}{2}\right) - 2^{n-1}\hat{U}_{n-1}\left(\frac{2-x}{2}\right) = (-1)^n \prod_{\nu=0}^{n-1}\left(x - 4\sin^2\left[\frac{(2\nu+1)}{2n+1}\frac{\pi}{2}\right]\right), \quad n \geqslant 1.$$

Hence (3.13) holds. □

Furthermore, we need the following identities which can be derived in a straightforward way from (3.11), (3.12), and (3.13).

$$\hat{J}_n\left(\frac{x-2}{2}; -\frac{1}{2}, \frac{1}{2}\right) = \frac{\cos\left(\left(n+\frac{1}{2}\right)z\right)}{(-1)^n\cos\left(\frac{z}{2}\right)} = \prod_{\nu=0}^{n-1}\left(x - 4\sin^2\left[\frac{(2\nu+1)}{2n+1}\frac{\pi}{2}\right]\right), \tag{4.14}$$

for any integer $n \geqslant 1$, where $x = 2(1 - \cos(z))$.

Next, we evaluate $\det\left[\sum_{\nu=0}^{l+k} C_{l+k-\nu}x^{\nu}\right]_{l,k=0}^{n}$ for some special values of x.

• For $x=0$ and $z=0$ in (3.12) and (3.13), we find

$$\det\left[C_{l+k}\right]_{l,k=0}^{n} = 2^n \prod_{\nu=0}^{n-1} \sin\left[\frac{(2\nu+1)}{2n+1}\frac{\pi}{2}\right] = 1.$$

• For $x=1$ and $z=\frac{\pi}{3}$ in (3.12) and (3.13),

$$\det\left[\sum_{\nu=0}^{l+k} C_{l+k-\nu}\right]_{l,k=0}^{n} = \prod_{\nu=0}^{n-1}\left(1-2\cos\left[\frac{(2\nu+1)\pi}{2n+1}\right]\right) = \begin{cases} 1, & \text{if } n\equiv 0,5 \pmod 6 \\ 0, & \text{if } n\equiv 1,4 \pmod 6 \\ -1, & \text{if } n\equiv 2,3 \pmod 6. \end{cases}$$

• For $x=2$ and $z=\frac{\pi}{2}$ in (3.12) and (3.13),

$$\det\left[\sum_{\nu=0}^{l+k} C_{l+k-\nu}2^{\nu}\right]_{l,k=0}^{n} = (-2)^n \prod_{\nu=0}^{n-1} \cos\left[\frac{(2\nu+1)\pi}{2n+1}\right] = (-1)^{\frac{n(n+1)}{2}}.$$

• For $x=3$ and $z=\frac{2\pi}{3}$ in (3.12) and (3.13),

$$\det\left[\sum_{\nu=0}^{l+k} C_{l+k-\nu}3^{\nu}\right]_{l,k=0}^{n} = (-1)^n \prod_{\nu=0}^{n-1}\left(1+2\cos\left[\frac{(2\nu+1)\pi}{2n+1}\right]\right)$$
$$= \begin{cases} 1, & \text{if } n\equiv 0,2 \pmod 3 \\ -2, & \text{if } n\equiv 1 \pmod 3. \end{cases}$$

• For $x=4$ and $z=\pi$ in (3.12) and (3.13),

$$\det\left[\sum_{\nu=0}^{l+k} C_{l+k-\nu}4^{\nu}\right]_{l,k=0}^{n} = (-1)^n(2n+1)$$
$$\prod_{\nu=0}^{n-1} \cos\left[\frac{(2\nu+1)}{2n+1}\frac{\pi}{2}\right] = 2^{-n}\sqrt{2n+1}.$$

Moreover, since (see [11]) $\sum_{\nu=0}^{n} C_{n-\nu}4^{\nu} = 2^{2n+1} - \binom{2n+1}{n+1},\ n\geqslant 0$, we obtain

$$\det\left[2^{2(l+k)+1} - \binom{2(l+k)+1}{l+k+1}\right]_{l,k=0}^{n} = (-1)^n(2n+1).$$

• For $x=2-\sqrt{2}$ and $z=\frac{\pi}{4}$ in (3.12) and (3.13),

$$\det\left[\sum_{\nu=0}^{l+k} C_{l+k-\nu}(2-\sqrt{2})^{\nu}\right]_{l,k=0}^{n} = (-1)^n \prod_{\nu=0}^{n-1}\left(2\cos\left[\frac{(2\nu+1)\pi}{2n+1}\right]-\sqrt{2}\right)$$
$$= \begin{cases} 1, & \text{if } n\equiv 0,7 \pmod 8 \\ -1, & \text{if } n\equiv 3,4 \pmod 8 \\ \sqrt{2}-1, & \text{if } n\equiv 1,6 \pmod 8 \\ 1-\sqrt{2}, & \text{if } n\equiv 2,5 \pmod 8. \end{cases}$$

4.2. Evaluation of $\det[\sum_{\nu=0}^{l+k+1} a_{l+k+1-\nu}x^{\nu}]_{l,k=0}^{n}$

Let us take $r=1$ in Corollary 1 and let us put $\hat{H}_0(\binom{A}{1}|\,x)=1$ and $\hat{H}_1(\binom{A}{1}|\,x)=x+a_0^{-1}a_1$. Then the SMP $(\hat{H}_n(\binom{A}{1}|\,x))_{n\geqslant 0}$ satisfies the following TTRR,

$$\begin{cases}\hat{H}_0(\binom{A}{1}|\,x)=1, \quad \hat{H}_1(\binom{A}{1}|\,x)=x+a_0^{-1}a_1,\\ \hat{H}_2(\binom{A}{1}|\,x)=(x-\alpha_0^{(2)})\hat{H}_1(\binom{A}{1}|\,x)+a_0^{-1}a_2\hat{H}_0(\binom{A}{1}|\,x),\\ \hat{H}_{n+2}(\binom{A}{1}|\,x)=(x-\alpha_n^{(2)})\hat{H}_{n+1}(\binom{A}{1}|\,x)-\lambda_n^{(2)}\hat{H}_n(\binom{A}{1}|\,x), \quad n\geqslant 1,\end{cases} \tag{4.15}$$

If the linear functional $x^2\mathscr{U}$ is quasi-definite then $a_2=(x^2\mathscr{U})_0\neq 0$. Thus, $(\hat{H}_n(\binom{A}{1}|\,x))_{n\geqslant 0}$ is a SMOP.

On the other hand, if we assume that the linear functionals $x^{\nu}\mathscr{U}$, $\nu=1,2$, are quasi-definite, another expression of the Hankel determinant $\hat{H}_n(\binom{A}{1}|\,x)$ in terms of the polynomials $\hat{H}_n(A\mid x)$ can be evaluated.

In this case, the corresponding SMOP $(\hat{Q}_n(x,A^{(\nu)}))_{n\geqslant 0}$, $\nu=1,2$, satisfies the following connection relation (see [9])

$$x\hat{Q}_n(x,A^{(2)})=\hat{Q}_{n+1}(x,A^{(1)})-\frac{\hat{Q}_{n+1}(0,A^{(1)})}{\hat{Q}_n(0,A^{(1)})}\hat{Q}_n(x,A^{(1)}), \quad n\geqslant 0. \tag{4.16}$$

From (2.8) with $r=1$ and (3.16), we get

$$\begin{aligned}\hat{H}_{n+1}(\binom{A}{1}|\,x)&=a_0^{-1}\mathscr{U}x\hat{Q}_n(x;A^{(2)})\\ &=a_0^{-1}\mathscr{U}\hat{Q}_{n+1}(x;A^{(1)})-\frac{\hat{Q}_{n+1}(0,A^{(1)})}{\hat{Q}_n(0,A^{(1)})}a_0^{-1}\mathscr{U}\hat{Q}_n(x;A^{(1)}), \quad n\geqslant 1,\end{aligned}$$

and, again, from (2.8) with $r=0$, we deduce

$$\hat{H}_{n+1}(\binom{A}{1}|\,x)=\hat{H}_{n+1}(A\mid x)-\frac{\hat{Q}_{n+1}(0,A^{(1)})}{\hat{Q}_n(0,A^{(1)})}\hat{H}_n(A\mid x), \quad n\geqslant 1. \tag{4.17}$$

As a consequence of Corollary 2, (2.8), and (3.17), we get

Corollary 3. *Let* $A=(a_n)_{n\geqslant 0}$ *be a sequence of complex numbers and let* $\mathscr{U}\in\mathbb{P}'$ *be such that* $(\mathscr{U})_n=a_n$, $n\geqslant 0$. *Suppose that* $a_0\neq 0$, *and* $x^{r+1}\mathscr{U}$, $r=0,1$, *are quasi-definite linear functionals. Then, we have*

$$\det\left[\sum_{\nu=0}^{l+k+1} a_{l+k+1-\nu}x^{\nu}\right]_{l,k=0}^{n}=\vartheta_n(\hat{H}_{n+1}(A\mid x)+\xi_n\hat{H}_n(A\mid x)), \quad n\geqslant 1,$$

where

$$\begin{aligned}&\vartheta_n=(-1)^n a_0\det[a_{l+k+2}]_{l,k=0}^{n-1}, \quad \xi_n=-\frac{\hat{Q}_{n+1}(0,A^{(1)})}{\hat{Q}_n(0,A^{(1)})},\\ &\hat{H}_n(A\mid x)=\hat{Q}_n(x,A^{(1)})+a_1a_0^{-1}\hat{Q}_{n-1}^{(1)}(x,A^{(1)}),\end{aligned}$$

and $\{\hat{Q}_n(x,A^{(1)})\}_{n\geqslant 0}$ *is the SMOP with respect to* $x\mathscr{U}$.

From Corollary 3, with $A=C$, and using (3.11) and (3.14), we next evaluate $\det\left[\sum_{\nu=0}^{l+k+1}\frac{1}{l+k+2-\nu}\binom{2(l+k+1-\nu)}{l+k+1-\nu}x^{\nu}\right]_{l,k=0}^{n}$.

Indeed, from (3.7) with $x=0$ and using induction, it is easy to show that $\hat{Q}_n(0,C^{(1)})=(-1)^n(n+1)$, $n\geqslant 0$. As a consequence, $\xi_n=\frac{n+2}{n+1}$, $n\geqslant 0$. On the other hand, let us remind that $\det[C_{l+k+2}]_{l,k=0}^{n-1}$

$= n+1, n \geqslant 1$, (see [4]). Therefore, taking into account (3.11) ,

$$\det\left[\sum_{\nu=0}^{l+k+1} C_{l+k+1-\nu}x^{\nu}\right]_{l,k=0}^{n} = (-2)^n(n+1)\left[2\hat{J}_{n+1}\left(\frac{x-2}{2};-\frac{1}{2},\frac{1}{2}\right) + \frac{n+2}{n+1}\hat{J}_n\left(\frac{x-2}{2};-\frac{1}{2},\frac{1}{2}\right)\right], \quad n \geqslant 1. \tag{4.18}$$

Substituting (3.14) into (3.18), we get

$$\det\left[\sum_{\nu=0}^{l+k+1} C_{l+k+1-\nu}x^{\nu}\right]_{l,k=0}^{n} = \frac{(n+2)\cos\left(\left(n+\frac{1}{2}\right)z\right) - (n+1)\cos\left(\left(n+\frac{3}{2}\right)z\right)}{\cos\left(\frac{z}{2}\right)}, \quad n \geqslant 1, \tag{4.19}$$

where $x = 2(1-\cos(z))$.

Next, we evaluate $\det\left[\sum_{\nu=0}^{l+k+1} C_{l+k+1-\nu}x^{\nu}\right]_{l,k=0}^{n}$ at the same values of x as above.

- For $x = 0$ and $z = 0$ in (3.19), we get

$$\det\left[C_{l+k+1}\right]_{l,k=0}^{n} = 1.$$

- For $x = 1$ and $z = \frac{\pi}{3}$ in (3.19),

$$\det\left[\sum_{\nu=0}^{l+k+1} C_{l+k+1-\nu}\right]_{l,k=0}^{n} = \begin{cases} n+2, & \text{if } n \equiv 0 \pmod 6 \\ n+1, & \text{if } n \equiv 1 \pmod 6 \\ -1, & \text{if } n \equiv 2 \pmod 6 \\ -(n+2), & \text{if } n \equiv 3 \pmod 6 \\ -(n+1), & \text{if } n \equiv 4 \pmod 6 \\ 1, & \text{if } n \equiv 5 \pmod 6). \end{cases}$$

- For $x = 2$ and $z = \frac{\pi}{2}$ in (3.19),

$$\det\left[\sum_{\nu=0}^{l+k+1} C_{l+k+1-\nu}2^{\nu}\right]_{l,k=0}^{n} = (-1)^{\frac{n(n+1)}{2}}\left[n+2+(-1)^n(n+1)\right].$$

- For $x = 3$ and $z = \frac{2\pi}{3}$ in (3.19),

$$\det\left[\sum_{\nu=0}^{l+k+1} C_{l+k+1-\nu}3^{\nu}\right]_{l,k=0}^{n} = \begin{cases} 3n+4, & \text{if } n \equiv 0 \pmod 3 \\ -3n-5, & \text{if } n \equiv 1 \pmod 3 \\ 1, & \text{if } n \equiv 2 \pmod 3). \end{cases}$$

- For $x = 4$ and $z = \pi$ in (3.19),

$$\det\left[\sum_{\nu=0}^{l+k+1} C_{l+k+1-\nu}4^{\nu}\right]_{l,k=0}^{n} = (-1)^n(4n^2+10n+5).$$

Since $\sum_{\nu=0}^{n+1} C_{n+1-\nu}4^{\nu} = 2^{2n+3} - \binom{2n+3}{n+2}, \quad n \geqslant 0$, then

$$\det\left[2^{2(l+k)+3} - \binom{2(l+k)+3}{l+k+2}\right]_{l,k=0}^{n} = (-1)^n(4n^2+10n+5).$$

- For $x = 2-\sqrt{2}$ and $z = \frac{\pi}{4}$ in (3.19),

$$\det\left[\sum_{\nu=0}^{l+k+1} C_{l+k+1-\nu}(2-\sqrt{2})^{\nu}\right]_{l,k=0}^{n} = \begin{cases} n(2-\sqrt{2})+3-\sqrt{2}, & \text{if } n \equiv 0 \pmod 8 \\ (\sqrt{2}-1)(2n+3), & \text{if } n \equiv 1 \pmod 8 \\ (2-\sqrt{2})n+3-2\sqrt{2}, & \text{if } n \equiv 2 \pmod 8 \\ -1, & \text{if } n \equiv 3 \pmod 8 \\ -n\sqrt{2}-1-\sqrt{2}, & \text{if } n \equiv 4 \pmod 8 \\ (1-\sqrt{2})(2n+3), & \text{if } n \equiv 5 \pmod 8 \\ (\sqrt{2}-2)n+2\sqrt{2}-3, & \text{if } n \equiv 6 \pmod 8 \\ 1, & \text{if } n \equiv 7 \pmod 8). \end{cases}$$

Acknowledgements

We thank the comments and suggestions by the referee which have contributed to improve the presentation of the manuscript. The work of the second author (F.M.) was supported by Dirección General de Investigación, Ministerio de Ciencia e Innovación of Spain, under grant MTM 2009-12740-C03-01. The work of the third author (R.S.) was supported by Faculté des Sciences de Gabès and Faculté des Sciences de Sfax, Tunisia.

References

[1] A.T. Benjamin, N.T. Cameron, J.J. Quinn, C.R. Yerger, Catalan determinants. A combinatorial approach, Congr. Numer. 200 (2010) 27–34.
[2] M.M. Chawla, On the Chebyshev polynomials of the second kind, SIAM Rev. 9 (1967) 729–733.
[3] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.
[4] J. Cigler, C.F. Krattenthaler, Some determinants of path generating functions, Adv. Appl. Math. 46 (2011) 144–174.
[5] C.F. Krattenthaler, Advanced determinant calculus, Sém. Lothar. Combin. 42 (1999) 67 (The Andrews Festschrift, Maratea, 1998), Article B42q.
[6] C.F. Krattenthaler, Advanced determinant calculus: a complement, Linear Algebra Appl. 411 (2005) 68–166.
[7] J.W. Layman, The Hankel transform and some of its properties, J. Integer Seq. 4 (2001) 11, Article 01.1.5.
[8] F. Marcellán, R. Sfaxi, A characterization of weakly regular linear functionals, Rev. Acad. Colombiana Cienc. Exact. Fis. Natur. 31 (119) (2007) 285–295.
[9] P. Maroni, Une théorie algébrique des polynômes orthogonaux. Applications aux polynômes orthogonaux semi-classiques, in: C. Brezinski (Ed.), et al., Orthogonal Polynomials and their Applications, Annals Comput. Appl. Math., vol. 9, Baltzer, Basel, 1991, pp. 95–130.
[10] P. Maroni, Fonctions eulériennes, Polynômes orthogonaux classiques, Tech. l'ingénieur A 154 (1994) 1–30.
[11] Ö. Egecioglu, A Catalan–Hankel determinant, Congr. Numer. 195 (2009) 49–63.
[12] Ö. Egecioglu, T. Redmond, C. Ryavec, A multilinear operator for almost product evaluation of Hankel determinants, J. Combin. Theory Ser. A 117 (2010) 77–103.